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Fitting Sampling Distribution Agreeing in Support and Moments and Tables of Critical Values of Sphericity Criterion

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Mixtures of two beta or gamma distributions having given first four, three, or two moments are found and applied to tabulate to three decimals critical values of the locally most powerful criterion of sphericity or proportions of moments of order two of a p-variate normal distribution. Expressions are given for the first four moments of the test statistic.

1. INTRODUCTION

It happens often that all that can be found about a sampling distribution are the first few moments. The moments are exact, not estimates as in estimation by method of moments. A Pearson curve is a frequency curve agreeing in the first four moments, which, for example, is applied in [3] to likelihood ratio test of sphericity of a normal distribution and other tests. Better fit may be expected if there is agreement in support also. We take the support to be either the unit interval or the right half line; if originally this is not the case, a function of the statistic may be found satisfying this condition. Agreement in one more moment may be achieved by raising the statistic to a suitable power, but we shall not consider this. Likelihood ratios satisfy our assumption about support naturally, and asymptotic results suggest a beta approximation to the sampling distribution of the statistic or a gamma approximation to the distribution of the logarithm of its reciprocal. We shall see how to fit a weighted average of two beta densities or of two gamma densities. One virtue both have is easy integrability. It has been possible to improve tabulations of many statistics by these methods.

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2. Fitting Sampling Distribution when Values of Statistic Lie between Zero and One

Let T be the statistic and $\mu_r = ET^r$, $r = 1, 2, \ldots$. If

$$a_0 = \mu_2 - \mu_1^2, \qquad a_1 = \mu_2 - \mu_1, \qquad b_0 = \mu_3 - \mu_1\mu_2 - 2a_0\mu_1,$$

$$b_1 = 3(\mu_3 - \mu_1\mu_2 - a_0), \qquad b_2 = 2(\mu_3 - \mu_2) - a_1,$$

$$c_0 = \mu_4 - 4\mu_3\mu_1 + 6\mu_2\mu_1^2 - 3\mu_1^4,$$

$$c_1 = 6(\mu_4 - 2\mu_1\mu_3 + \mu_1^2\mu_2 - \mu_3 + 2\mu_1\mu_2 - \mu_1^3),$$

$$c_2 = 11\mu_4 - 8\mu_3\mu_1 - 18\mu_3 + 12\mu_2\mu_1 + 7\mu_2 - 4\mu_1^2,$$

$$c_3 = 6\mu_4 - 12\mu_3 + 7\mu_2 - \mu_1,$$

$$(a_0c_0 - a_0^3 - b_0^2) D^3 + (a_0c_1 + a_1c_0 + a_0^3 + b_0^2 - a_0c_0 - 2b_0b_1 - 3a_0^2a_1) D^2 + (a_0c_3 + a_1c_2 + b_2^2 - a_1^3 - a_1c_3 - 2b_1b_2) D + a_1c_3 - b_2^2 = 0,$$

$$\delta_1 = (a_0D + a_1)D, \ \delta_2 = \tfrac{1}{2}(b_0D^2 + b_1D + b_2)\,D\delta_1^{-1},$$

$$d_1 = D\mu_1 + \delta_2 - \{\delta_1 + \delta_2^2\}^{1/2}, \ d_2 = D\mu_1 + \delta_2 + \{\delta_1 + \delta_2^2\}^{1/2},$$

and

$$W = (d_2 - D\mu_1)(d_2 - d_1)^{-1},$$

$$\mu_r = \int_0^1 t^r \left\{ W \frac{\Gamma(D)}{\Gamma(d_1)\,\Gamma(D-d_1)}\, t^{d_1-1}(1-t)^{D-d_1-1} + (1-W)\frac{\Gamma(D)}{\Gamma(d_2)\,\Gamma(D-d_2)}\, t^{d_2-1}(1-t)^{D-d_2-1} \right\} dt, \tag{2.1}$$

$r = 1, 2, 3, 4$, so that the density of T at t, $0 \leqslant t \leqslant 1$, is approximately

$$W \frac{\Gamma(D)}{\Gamma(d_1)\,\Gamma(D-d_1)}\, t^{d_1-1}(1-t)^{D-d_1-1} + (1-W)\frac{\Gamma(D)}{\Gamma(d_2)\,\Gamma(D-d_2)}\, t^{d_2-1}(1-t)^{D-d_2-1}. \tag{2.2}$$

The conditions $D > d_i > 0$, $i = 1, 2$, $0 \leqslant W \leqslant 1$ must be satisfied. If the solution is not unique, choose the one for which

$$\left| \left\{ W \frac{\Gamma(d_1+5)\,\Gamma(D)}{\Gamma(d_1)\,\Gamma(D+5)} + (1-W)\frac{\Gamma(d_2+5)\,\Gamma(D)}{\Gamma(d_2)\,\Gamma(D+5)} \right\} \mu_5^{-1} - 1 \right|$$

is least. This situation did not arise in the application to the sphericity criterion.

If d_1 can be fixed from other considerations, and $(b_0\mu_1 + a_0\mu_1^2 - a_0^2) D^3 + (b_1\mu_1 + a_1\mu_1^2 - b_0 d_1 - 2a_0a_1 - 2a_0d_1\mu_1)D^2 + (b_2\mu_1 + a_0d_1^2 - a_1^2 - b_1d_1 - 2a_1d_1\mu_1)D + a_1d_1^2 - b_2d_1 = 0$, $W = \{(D\mu_1 - d_1)^2 + D(a_0D + a_1)\}^{-1}(a_0D + a_1) D$, $d_2 = (1 - W)^{-1}(D\mu_1 - Wd_1)$, Eq. (2.1) is satisfied for $r = 1, 2, 3$, so that the density of T at t is approximately (2.2). If the solution is not unique, choose the one for which

$$\left|\left\{W \frac{\Gamma(d_1+4)\,\Gamma(D)}{\Gamma(d_1)\,\Gamma(D+4)} + (1-W)\frac{\Gamma(d_2+4)\,\Gamma(D)}{\Gamma(d_2)\,\Gamma(D+4)}\right\} \mu_4^{-1} - 1\right|$$

is least. In the application to the sphericity criterion d_1 could be fixed as $\frac{1}{4}p(p+1) - \frac{1}{2}$, where p is the dimension of the normal distribution whose sphericity is to be tested, from the fact that the asymptotic distribution of T is $\chi^2_{(1/2)p(p+1)-1}$; the consequent loss in accuracy would have been quite small.

If the last method also fails, take the distribution of T to be approximately beta with degrees of freedom $(\mu_1^2 - \mu_2)^{-1}(\mu_2 - \mu_1)\mu_1$ and $(\mu_1^2 - \mu_2)^{-1}(\mu_2 - \mu_1)(1 - \mu_1)$, whose first two moments are the same as those of T. In the application to the sphericity criterion the first method never failed.

3. Fitting of Sampling Distribution when Values of Statistic Lie between Zero and Infinity

If

$$a = \mu_2 - \mu_1 - \mu_1^2, \qquad b = \mu_3 - 3\mu_2 + \mu_1 - 3\mu_1\mu_2 + 3\mu_1^2 + 2\mu_1^3,$$
$$W = \tfrac{1}{2} \pm (4a^3 + b^2)^{-1/2} b, \qquad d_1 = \mu_1 - \{a(W^{-1} - 1)\}^{1/2},$$

and

$$d_2 = \mu_1 + \{(1-W)^{-1}Wa\}^{1/2},$$
$$\mu_r = \int_0^\infty t^r[W\{\Gamma(d_1)\}^{-1}t^{d_1-1}e^{-t} + (1-W)\{\Gamma(d_2)\}^{-1}t^{d_2-1}e^{-t}]\,dt, \tag{3.1}$$

$r = 1, 2, 3$, so that the density of T at t, $0 \leqslant t < \infty$, is approximately

$$W\{\Gamma(d_1)\}^{-1}t^{d_1-1}e^{-t} + (1-W)\{\Gamma(d_2)\}^{-1}t^{d_2-1}e^{-t}. \tag{3.2}$$

The conditions $d_i > 0$, $i = 1, 2$, $0 \leqslant W \leqslant 1$ must be satisfied. If the solution is not unique, choose the one for which

$$\left|\left\{W \frac{\Gamma(d_1+4)}{\Gamma(d_1)} + (1-W)\frac{\Gamma(d_2+4)}{\Gamma(d_2)}\right\} \mu_4^{-1} - 1\right|$$

is least. Agreement in one more moment may be achieved by replacing T by cT and adjusting c, but we shall not consider this.

If d_1 can be fixed from other considerations, and

$$W = \{(\mu_1 - d_1)^2 + \mu_2 - \mu_1 - \mu_1^2\}^{-1}(\mu_2 - \mu_1 - \mu_1^2)$$

and

$$d_2 = (1 - W)^{-1}(\mu_1 - Wd_1),$$

Eq. (3.1) is satisfied for $r = 1, 2$, so that the density of T at t is approximately (3.2). Alternatively, there would be agreement in the first two moments, also if we take the distribution of $(\mu_2 - \mu_1^2)^{-1}\mu_1 T$ to be gamma with degree of freedom $(\mu_2 - \mu_1^2)^{-1}\mu_1^2$. The goodness of the approximations should be investigated in any particular application.

4. Support and Moments of Sphericity Criterion

The locally most powerful test of sphericity or more generally proportionality of the covariance matrix of a p-variate normal distribution to a given positive definite $\mathbf{\Sigma}_0$ is to reject the hypothesis (H_0), if $U = (\text{tr}\, \mathbf{\Sigma}_0^{-1}\mathbf{S})^{-2}\, \text{tr}\, (\mathbf{\Sigma}_0^{-1}\mathbf{S})^2$, where $\mathbf{S}$ is the matrix of corrected sums of squares and products of the $N = n + 1$ random observations on the p variates, is too large; see [1]. John [2] shows that the support of $T = (1 - p^{-1})^{-1}(U - p^{-1})$ is $[0, 1]$. We shall take $\mathbf{\Sigma}_0 = \mathbf{I}$; this will not affect the distribution of T under H_0.

Let $\kappa_j(n, p) = n^{-2j}E\{(\text{tr}\, S^2)^j \mid H_0\}$. Then

$$\begin{aligned} E(U^j \mid H_0) &= \{(p)_{2j,n}\}^{-1}\kappa_j(n, p), \\ &= \lambda_j \text{ (say)}, \end{aligned}$$

if $(x)_{m,q} = x(x + 2q^{-1})(x + 4q^{-1}) \cdots \{x + (2m - 2)q^{-1}\}$, and

$$\mu_r = (1 - p^{-1})^{-r} \sum_{j=0}^{r} \binom{r}{j} (-p^{-1})^{r-j}\lambda_j, \qquad r = 1, 2, \ldots; \qquad \text{see [2].}$$

John [2] gives

$$\kappa_1(n, p) = p\{1 + (p + 1)n^{-1}\},$$

$$\begin{aligned} \kappa_2(n, p) = {} & p^2 + (2p^3 + 2p^2 + 8p)n^{-1} + (p^4 + 2p^3 + 21p^2 + 20p)n^{-2} \\ & + (8p^3 + 20p^2 + 20p)n^{-3}. \end{aligned}$$

To obtain recurrence relations for $\kappa_j(n, p)$, $j > 2$, write

$$\operatorname{tr} \mathbf{S}^2 = \sum_{i=1}^{p-1} \sum_{j=1}^{p-1} S_{ij}^2 + 2 \sum_{i=1}^{p-1} S_{ip}^2 + S_{pp}^2 ,$$

where the S_{ij}'s are the elements of $\mathbf{S}$, and use the expansion of powers of trinomials before taking expectations. Two illustrative calculations follow.

$$E\left(\sum_{i=1}^{p-1} \sum_{j=1}^{p-1} S_{ij}^2\right)^2 \sum_{i=1}^{p-1} S_{ip}^2 = E\left\{E\left(\sum_{i=1}^{p-1} \sum_{j=1}^{p-1} S_{ij}^2\right)^2 \sum_{i=1}^{p-1} S_{ip}^2 \mid S_{ij}^2 , i, j = 1,..., p-1\right\},$$

$$= E\left\{\left(\sum_{i=1}^{p-1} \sum_{j=1}^{p-1} S_{ij}^2\right)^2 \sum_{i=1}^{p-1} S_{ii}\right\},$$

$$= \left\{E\left(\sum_{i=1}^{p-1} S_{ii}\right)^{-4} \left(\sum_{i=1}^{p-1} \sum_{j=1}^{p-1} S_{ij}^2\right)^2\right\} E\left(\sum_{i=1}^{p-1} S_{ii}\right)^5 \quad \text{by [2]},$$

$$= n^4(np - n + 4)\, \kappa_2(n, p-1).$$

$$E \sum_{i=1}^{p-1} \sum_{j=1}^{p-1} S_{ij}^2 \left(\sum_{i=1}^{p-1} S_{ip}^2\right)^3 = E \sum_{i=1}^{p-1} \sum_{j=1}^{p-1} (V_{ij} + y_i y_j)^2 \left(\sum_{i=1}^{p-1} y_i^2\right)^3 S_{pp}^3 ,$$

where $y_i = S_{pp}^{-1/2} S_{pi}$ and

$$V_{ij} = S_{ij} - y_i y_j , \; i, j = 1,..., p-1,$$

$$= E\left\{\left(\sum_{i=1}^{p-1} \sum_{j=1}^{p-1} V_{ij}^2\right)\left(\sum_{i=1}^{p-1} y_i^2\right)^3 + 2(n-1)\left(\sum_{i=1}^{p-1} y_i^2\right)^4 + \left(\sum_{i=1}^{p-1} y_i^2\right)^5\right\} S_{pp}^3 ,$$

$$= [(n-1)^3(p-1)\{(n-1)(p-1)+1\}\, \kappa_1(n-1, p-1)$$
$$+ 2(n-1)(p+2) + (p+2)(p+3)]\,(p-1)_3 (n)_3 ,$$

where $(n)_j = n(n+1) \cdots (n+j-1)$. The recurrence relations for $\kappa_3(n, p)$ and $\kappa_4(n, p)$ follow.

$$\begin{aligned}\kappa_3(n, p) = {} & \kappa_3(n, p-1) + 3\{1 + 2(p + 12n^{-1})\, n^{-1}\}\, \kappa_2(n, p-1) \\ & + 3[(1)_{4,n} + 4(p-1+4n^{-1})\{1 + (p+5)n^{-1} + 14n^{-2}\}n^{-1}]\kappa_1(n, p-1) \\ & + (1)_{6,n} + 6(p-1)(1)_{5,n} n^{-1} \\ & + 12(p^2-1)(1)_{4,n} n^{-2} + 8(p-1)_{3,1}(1)_{3,n} n^{-3};\end{aligned}$$

$$\begin{aligned}\kappa_4(n,p) = {} & \kappa_4(n,p-1) + 4(1 + 2pn^{-1} + 36n^{-2})\,\kappa_3(n,p-1) \\ & + 6[\{1 + 16((1)_{2,n})^{-1}n^{-2}\}(1)_{4,n} + 4(p-1+8n^{-1}) \\ & \times \{(1)_{3,n} + (p-1+10n^{-1})n^{-1}\}n^{-1}]\,\kappa_2(n,p-1) \\ & + 4[(1)_{6,n} + 6(p-1+4n^{-1}) \\ & \times \{(1)_{5,n} + 2(p-1+6n^{-1})(1)_{4,n}((1)_{2,n})^{-1}n^{-1}\}n^{-1}]\,\kappa_1(n,p-1) \\ & + 32(p-1)_{3,1}(1-n^{-1})^2(1)_{3,n}n^{-3}\kappa_1(n-1,p-1) + (1)_{8,n} \\ & + 8(p-1)(1)_{7,n}n^{-1} + 24(p^2-1)(1)_{6,n}n^{-2} + 32(p-1)_{3,1}(1)_{5,n}n^{-3} \\ & + 16(5+2n^{-1})(p-1)_{4,1}(1)_{3,n}n^{-4} + 32(p-1)_{5,1}(1)_{3,n}n^{-5}.\end{aligned}$$

5. The Tables

For interpolation and extrapolation use n for $n \leqslant 25$ and n^{-1} for $n > 25$. For $n = \infty$ all critical values may be taken to be zero. Comparison of the approximations with the exact values for $p = 3$ show that the approximations differ very little from exact values. The behaviour of skewness and kurtosis improves with increase of p. Tables for $p = 2$ are omitted because for $p = 2$ the distribution of T is exactly beta; see [2]. Entries for $n < p$ are omitted because if $n < p$ the distribution of U is the same as the distribution of U with n and p interchanged. The tables included are a selection from tables covering more n's, p's and significance levels. Fitting parameter values were also tabulated.

TABLE I

Exact Critical Values of T in the Trivariate Case

	Significance level					Significance level			
n	0.1	0.05	0.01	0.001	n	0.1	0.05	0.01	0.001
3	0.740	0.816	0.917	0.974	13	0.216	0.256	0.342	0.451
4	0.600	0.681	0.813	0.913	14	0.202	0.239	0.321	0.424
5	0.502	0.579	0.716	0.839	15	0.189	0.224	0.302	0.400
6	0.431	0.501	0.635	0.768	16	0.178	0.212	0.285	0.379
7	0.377	0.442	0.568	0.703	18	0.160	0.190	0.256	0.342
8	0.335	0.394	0.513	0.646	20	0.145	0.172	0.232	0.312
9	0.302	0.356	0.467	0.596	22	0.132	0.157	0.212	0.287
10	0.274	0.324	0.429	0.552	24	0.122	0.145	0.196	0.265
11	0.252	0.298	0.396	0.514	180	0.017	0.020	0.028	0.038
12	0.232	0.275	0.367	0.480					

TABLE II

Five Percent Critical Values of T

p / n	4	5	6	7	8	9	10
4	0.585						
5	0.482	0.425					
6	0.410	0.359	0.326				
7	0.357	0.311	0.282	0.261			
8	0.316	0.275	0.248	0.229	0.216		
9	0.284	0.246	0.222	0.205	0.192	0.183	
10	0.258	0.223	0.200	0.185	0.174	0.165	0.158
11	0.236	0.203	0.183	0.169	0.158	0.150	0.144
12	0.218	0.187	0.168	0.155	0.145	0.138	0.132
13	0.202	0.174	0.156	0.143	0.134	0.128	0.122
14	0.189	0.162	0.145	0.133	0.125	0.119	0.114
15	0.177	0.151	0.135	0.125	0.117	0.111	0.106
16	0.166	0.142	0.127	0.117	0.110	0.104	0.099
18	0.149	0.127	0.113	0.104	0.098	0.092	0.088
20	0.134	0.115	0.102	0.094	0.088	0.083	0.080
22	0.123	0.105	0.093	0.086	0.080	0.076	0.072
24	0.113	0.096	0.086	0.079	0.073	0.070	0.066
180	0.016	0.013	0.012	0.011	0.010	0.009	0.009

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